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Abstract

We investigate the influence of an electric field on trapped modes arising in a two-dimensional curved quantum waveguide $\mathbf{\Omega}$ i.e. bound states of the corresponding Laplace operator $-\Delta_{\mathbf{\Omega}}$. Here the curvature of the guide is supposed to satisfy some assumptions of analyticity, and decays as $O(|s|^{-\varepsilon}), \varepsilon > 3$ at infinity. We show that under conditions on the electric field $\mathbf{F}$, $\mathbf{H}(F) := -\Delta_{\mathbf{\Omega}} + \mathbf{F}.\mathbf{x}$ has resonances near the discrete eigenvalues of $-\Delta_{\mathbf{\Omega}}$.

1 Introduction

This paper is a continuation and extension of earlier work [2]. Let us recall the problem; for more details we refer to [2, 8]. The object of our interest is the Stark operator

$$\mathbf{H}(F) = -\Delta_{\mathbf{\Omega}} + \mathbf{F} \cdot \mathbf{x}, \ \mathbf{F} \in \mathbb{R}^2, \ \mathbf{x} = (x, y) \in \mathbf{\Omega} \tag{1.1}$$

on $L^2(\mathbf{\Omega})$ where $\mathbf{\Omega}$ is a curved strip in $\mathbb{R}^2$ of constant width $d > 0$ defined around a smooth curve Γ. The operator $-\Delta_{\mathbf{\Omega}}$ is defined in a standard way by means of DBC on the boundary of $\mathbf{\Omega}$, $\partial\mathbf{\Omega}$ [13].

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We assume that $\mathbf{\Omega}$ is not straight, let $s \in \mathbb{R} \to \gamma(s)$ be the signed curvature of Γ. In [2] it is supposed that $\gamma \in C_0^2(\mathbb{R})$ and $d\|\gamma\|_\infty < 1$. Here we consider a more general situation namely we assume that

(h1) $\gamma \in C^2(\mathbb{R})$ and there exist $a_0, r_0 > 0$ s.t. γ has an analytic extension in

$$\mathcal{O}_{a_0,r_0} = \{z \in \mathbb{C},\ |\arg z| < a_0\} \cup \{z \in \mathbb{C},\ |\pi - \arg z| < a_0\} \cap \{|\mathrm{Re} z| > r_0\}.$$

Moreover γ satisfies $d\|\mathrm{Re}\gamma\|_\infty < 1$.

(h2) There exists $\varepsilon > 3$ s.t. $\gamma(z) = O(|z|^{-\varepsilon})$ as $|\mathrm{Re} z| \to \infty$.

Then $\mathbf{\Omega}$ is asymptotically straight. We choose the lower boundary of $\mathbf{\Omega}$ near $s = -\infty$ as the reference curve. Introduce orthogonal coordinates $(s, u) \in \Omega := \mathbb{R} \times (0, d)$, related to $(x, y) \in \mathbf{\Omega}$ via the relations [6],

$$x(s,u) = \int_0^s \cos(\alpha(t))dt - u\sin(\alpha(s)),\ y(s,u) = \int_0^s \sin(\alpha(t))dt + u\cos(\alpha(s)) \quad (1.2)$$

where $\alpha(s) = \displaystyle\int_{-\infty}^s \gamma(t)\, dt$. Set $\alpha_0 = \displaystyle\int_{-\infty}^{+\infty} \gamma(t)\, dt$.

Since we have supposed $d\|\gamma\|_\infty < 1$, the operator $-\Delta_{\mathbf{\Omega}}$ is unitarily equivalent to

$$H = H_0 + V_0, H_0 = T_s + T_u \quad (1.3)$$

in the space $L^2(\Omega)$, with DBC on $\partial\Omega$ [6], where

$$T_s := -\partial_s g \partial_s,\ \ T_u := -\partial_u^2,\ \ g(s,u) = (1 + u\gamma(s))^{-2}, \quad (1.4)$$

and

$$V_0(s,u) = -\frac{\gamma(s)^2}{4(1+u\gamma(s))^2} + \frac{u\gamma''(s)}{2(1+u\gamma(s))^3} - \frac{5}{4}\frac{u^2\gamma'(s)^2}{(1+u\gamma(s))^4}. \quad (1.5)$$

With our assumptions, the potential V_0 is bounded and then $H = H_0 + V_0$ is a self-adjoint operator with domain [7, 12],

$$D(H) = D(H_0) = \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega). \quad (1.6)$$

Here and hereinafter we use standard notation for Sobolev space. Moreover the essential spectrum of this operator, $\sigma_{ess}(H) = [\lambda_0, +\infty)$, where $\{\lambda_0, \lambda_1, ...\}$ are the transverse modes of the system i.e. the eigenvalues of the operator $-\partial_u^2$ on $L^2(0, d)$ with DBC on the boundary $\{0, d\}$ [4].
Denote the exterior field as $\mathbf{F} = F(\cos(\eta), \sin(\eta))$. With respect to the new coordinates, the field interaction is then

$$W(F)(s,u) := \mathbf{F} \cdot \mathbf{x} = F\int_0^s \cos(\eta - \alpha(t))\, dt + Fu\sin(\eta - \alpha(s)). \quad (1.7)$$

Here we study a field regime which was not considered so far i.e. the intensity of the field is the free parameter in $0 < F < 1$ and the direction η is fixed satisfying

$$|\eta| < \frac{\pi}{2} \quad \text{and} \quad |\eta - \alpha_0| > \frac{\pi}{2}. \tag{1.8}$$

As discussed in the Section 2, this implies that $W(F)(s,u) \to -\infty$ as $s \to \pm\infty$. Thus the non trapping region for a given negative energy E contains both a neighbourhood of $s = -\infty$ and $s = \infty$.

We denote by $H_0(F) = T_s + T_u + W(F)$ and $H(F) = H_0(F) + V_0$. Then a straightforward extension of the Theorem 2.1 of [2] shows that for $F > 0$, the Stark operator $H(F)$ is essentially self-adjoint and $\sigma(H(F)) = \mathbb{R}$.

We are interested in the study of the influence of the electric field on the discrete spectrum of H. We want to show that the eigenvalues of H give rise to resonances for the Stark operator $H(F), F > 0$. The resonances of $H(F)$ are understood in the standard way [1, 10, 13]. Evidently if H has no discrete eigenvalue below λ_0 then this result proves that $H(F)$ has neither resonance or embedded eigenvalue in $\{z \in \mathbb{C}, \mathrm{Re} z < \lambda_0\}$. For a discussion about eigenvalues of H we refer the reader to [4].

To study this problem we need an additional assumption,

(h3) $\mathrm{Im}\gamma(z) \geq 0$ for $z \in \mathcal{O}_{a_0,r_0}, 0 \leq \arg z \leq a_0$ or $0 \leq \arg z - \pi \leq a_0$.

Remark 1.1. *i) In fact it is only necessary to suppose that the product* $u\mathrm{Im}\gamma(z) \geq 0$ *for* $z \in \mathcal{O}_{a_0,r_0}, 0 \leq \arg z \leq a_0$ *or* $0 \leq \arg z - \pi \leq a_0$. *So the case* $Im\gamma(z) \leq 0$ *is reducing to the present one by taking the other boundary as reference curve.*
ii) We can check that $\gamma(s) = \frac{\alpha}{1+s^{2n}}$; $n \geq 2$, $\alpha < 0$ *satisfies our assumptions with* $r_0 > 1$ *and* $0 < a_0 \leq \frac{\pi}{4n}$.

The study of the Stark effect was considered by several authors, see e.g. [3, 9] for a discussion concerning the case of Shrödinger operators on $\mathbb{R}^n$ and [2, 5] for operators defined on curved strips. In particular in [2], under assumptions on the curvature above mentioned and if $|\eta| < \frac{\pi}{2}$ and $|\eta - \alpha_0| < \frac{\pi}{2}$, it is proved the existence of Stark resonances having an width exponentially small w.r.t. F as F tends to zero. In this paper we would like to extend this result under weaker assumptions on γ i.e. hypotheses (h1-3) and in the field regime (1.8).
More precisely we will show the following.

Theorem 1.2. *Suppose (h1-3). Let* E_0 *be an discrete eigenvalue of* H *of finite multiplicity* $n \in \mathbb{N}$. *There exist* $F_0 > 0$ *and a dense subset* $\mathcal{A}$ *of* $L^2(\Omega)$ *such that for* $0 < F \leq F_0$

$$i) \quad z \in \mathbb{C}, \mathrm{Im} z > 0 \to \mathcal{R}_\varphi(z) = \big((H(F) - z)^{-1}\varphi, \varphi\big),\ \varphi \in \mathcal{A}$$

has an meromorphic extension in a complex neighbourhood ν_{E_0} *of* E_0*, through the cut due to the presence of the continuous spectrum of* $H(F)$*.*
ii) $\cup_{\varphi\in\mathcal{A}}\{\text{poles of } \mathcal{R}_\varphi(z)\}\cap\nu_{E_0}$ *contains* n *poles* $Z_0(F),...Z_{n-1}(F)$ *converging to* E_0 *when* $F\to 0$*.*
iii) Suppose that E_0 *is a simple eigenvalue of* H*, Let* $Z_0(F)$ *as in ii). For Then there exist two constants* $0<c_1,c_2$ *such that for* $0<F\le F_0$*,*

$$|\mathrm{Im} Z_0|\le c_1 e^{-\frac{c_2}{F}}.$$

In this paper we only give elements we need to extend the strategy of [2] to the situation we now consider. In particular Theorem 1.2 iii) is covered by [2, Section 6], so we omit the proof here.

The plan of this work is as follows. In section 2 we introduce a local modification of the operator $H(F)$ we need to perform the meromorphic continuation of the resolvent of $H(F)$. Some elements of the the complex distortion theory are given in the Section 3. In section 4 we define the extension of the resolvent of $H(F)$, this allows to define the resonances of $H(F)$. The existence of resonances is proved in the Section 6. The section 7 is devoted to some concluding remarks.

2 The reference operator

To prove the theorem 1.2 we use the distortion theory such that it can be found in [2, 3]. The first step is to consider a local modification of the operator $H_0(F)$ called the reference operator. It is defined as follow. Note that

$$\text{if } s<0,\ W(F,s,u)=F\big(s\cos(\eta)+u\sin(\eta)+A_-\big)+R_-(F,s), \tag{2.9}$$

where $R_-(F,s)=F\Big(\displaystyle\int_{-\infty}^{s}(\cos(\eta-\alpha(t))-\cos(\eta))dt+u(\sin(\eta-\alpha(s))-\sin(\eta))\Big)$.

$$\text{if } s\ge 0,\ W(F,s,u)=F\big(s\cos(\eta-\alpha_0)+u\sin(\eta-\alpha_0)+A_+\big)+R_+(F,s) \tag{2.10}$$

where $R_+(F,s)=F\Big(\displaystyle\int_{s}^{\infty}(\cos(\eta-\alpha_0)-\cos(\eta-\alpha(t))dt+u(\sin(\eta-\alpha(s))-\sin(\eta-\alpha_0))\Big)$.

The constants A_-,A_+ are

$$A_-:=\int_{-\infty}^{0}(\cos(\eta)-\cos(\eta-\alpha(t))dt,\ A_+:=\int_{0}^{\infty}(\cos(\eta-\alpha(t))-\cos(\eta-\alpha_0)dt.$$

In view of (h2) and

$$\alpha(s)=O(\frac{1}{|s|^{\varepsilon-1}}),\ \text{as } s\to-\infty;\ \alpha(s)=\alpha_0+O(\frac{1}{|s|^{\varepsilon-1}}),\ \text{as s}\to\infty, \tag{2.11}$$

A_-, A_+ are well defined. Moreover we have,

$$R_-(F,s), R_+(F,s) = O(\frac{F}{|s|^{\varepsilon-2}}), \quad \text{as } s \to \pm\infty. \tag{2.12}$$

Set $R_-(F,s) = 0$ for $s \geq 0$ and $R_+(F,s) = 0$ for $s < 0$.
Hence it is quite natural to consider a modified interaction defined as,

$$\tilde{W}(F)(s,u) = F(s\cos(\eta) + u\sin(\eta) + A_-) \text{ for } s < 0$$

and

$$\tilde{W}(F)(s,u) = F(s\cos(\eta-\alpha_0) + u\sin(\eta-\alpha_0) + A_+) \;\text{ for } s \geq 0.$$

In particular we get

$$W(F) - \tilde{W}(F) = R(F) := R_+(F) + R_-(F) = O(\frac{F}{|s|^{\varepsilon-2}}) \text{ as } s \to \pm\infty. \tag{2.13}$$

Notice also that

$$R'(F) = O(\frac{F}{|s|^{\varepsilon-1}}), \; R''(F) = O(\frac{F}{|s|^{\varepsilon}})\text{as } \; s \to \pm\infty. \tag{2.14}$$

Let $\tilde{H}_0(F)$ be the reference operator in $\mathrm{L}^2(\Omega)$,

$$\tilde{H}_0(F) := H_0 + \tilde{W}(F).$$

$\tilde{H}_0(F)$ differs from $H_0(F)$ by an additional bounded operator, then it is essentially self-adjoint and $\sigma(\tilde{H}_0(F)) = \mathbb{R}$.

3 Complex distortion

In this section, we give necessary elements for the complex distortion theory [1, 3, 10] we need to define the family of distorted operators $\{H_\theta(F), F \leq F_0\}$, for some $F_0 > 0$ and complex values of θ. We denote by $\beta = \mathrm{Im}\theta$.

Let $E \in \mathbb{R}$, $E < 0$ be the reference energy and $0 < \delta E < \frac{1}{2}\min\{1, |E|\}$. Denote $E_- = E - \delta E$ and $E_+ = E + \delta E$. We choose a real function $\phi \in \mathrm{C}^\infty(\mathbb{R})$ such that:

$$\phi(t) = \begin{cases} 1 & \text{if } t < E, \\ 0 & \text{if } t > E_+ \end{cases} \tag{3.15}$$

and satisfying $\|\phi^{(k)}\|_\infty = O((\frac{1}{\delta E})^k)$.
For $\theta \in \mathbb{R}$, we introduce the distortion on Ω, $s_\theta(s,u) := (s + \theta f(s), u)$ where

$$f(s) = \begin{cases} -\frac{1}{F cos(\eta)}\Phi_-(s) & \text{if } s \leq 0, \\ -\frac{1}{F\cos(\eta-\alpha_0)}\Phi_+(s) & \text{if } s > 0 \end{cases} \tag{3.16}$$

where $\Phi_-(s) = \phi\big(F\cos(\eta)s\big)$ and $\Phi_+(s) = \phi\big(F\cos(\eta-\alpha_0)s\big)$. Set $\Phi(s) = \Phi_-(s) + \Phi_+(s)$.

So for $k \geq 1$, $\|\Phi^{(k)}\|_\infty \leq \left(\frac{F}{\delta E}\right)^k$, $\quad \|f^{(k)}\|_\infty \leq \frac{F^{k-1}}{(\delta E)^k}$.

In view of the definition (3.16), for small F, s_θ is an translation along the longitudinal axis in neighbourood of $s = \pm\infty$ since

$$f(s) = -\frac{1}{F\cos(\eta)} \text{ for } s \leq \frac{E}{F\cos\eta} \text{ and } f(s) = -\frac{1}{F\cos(\eta-\alpha_0)} \text{ for } s \geq \frac{E}{\cos(\eta-\alpha_0)}. \tag{3.17}$$

Assume $|\theta| < \delta E$. Let U_θ be the operator defined on $L^2(\Omega)$ by

$$U_\theta\psi(s,u) = (1+\theta f')^{\frac{1}{2}}\psi\left(s_\theta(s,u)\right). \tag{3.18}$$

The operators U_θ are unitary and generate the family,

$$H_\theta(F) = U_\theta H(F) U_\theta^{-1} = H_{0,\theta}(F) + V_{0,\theta} \tag{3.19}$$

where

$$H_{0,\theta}(F) := T_{s,\theta} + T_u + W_\theta(F), \tag{3.20}$$

with

$$T_{s,\theta} = -(1+\theta f')^{-\frac{1}{2}}\partial_s(1+\theta f')^{-1}g_\theta\partial_s(1+\theta f')^{-\frac{1}{2}}, \tag{3.21}$$

$g_\theta = (1+u\gamma_\theta)^{-2}, \gamma_\theta = \gamma\circ s_\theta,\ W_\theta(F) = W(F)\circ s_\theta$ and $V_{0,\theta} := V_0 \circ s_\theta$.
We also have

$$T_{s,\theta} = -\partial_s(1+\theta f')^{-2}g_\theta\partial_s + S_\theta, \tag{3.22}$$

where

$$S_\theta = -\frac{5g_\theta}{4}\frac{\theta^2 f''^2}{(1+\theta f')^4} + \frac{g_\theta}{2}\frac{\theta f'''}{(1+\theta f')^3} + \frac{g'_\theta}{2}\frac{\theta f''}{(1+\theta f')^3}. \tag{3.23}$$

For small F,the analytic extension of the family $\{H_\theta(F), \theta \in \mathbb{R}, |\theta| < \delta E\}$ to a complex disk $|\theta| \leq \beta_0$ for some $\beta_0 > 0$ depends strongly on the analytic property of the dilated curvature γ_θ. But clearly (h1) and the definition (3.18) of s_θ imply that if $0 < F \leq F_0$ with F_0 is small enough, $s \in \mathbb{R}$, $\theta \to \gamma_\theta(s)$ is analytic in $|\theta| \leq a_0$.

Then the same arguments as in the proof of [2, Proposition (3.1)] lead to

Proposition 3.3. *Suppose (h1-2). There exist $F_0 > 0$ and $0 < \beta_0 \leq \min\{\delta E, a_0\}$ such that for all $0 < F \leq F_0$, $\{H_\theta(F); |\theta| \leq \beta_0\}$ is a self-adjoint analytic family of operators.*

It should be noted that all the critical values β_0 and F_0 that appear in this article are independent from each other.

In a similar way let $\tilde{H}_{0,\theta}(F) = H_{0,\theta} + \tilde{W}_\theta(F)$, where $\tilde{W}_\theta(F) = \tilde{W}_\theta(F) \circ s_\theta$. Since

$$R_\theta(F) = W_\theta(F) - \tilde{W}_\theta(F) = R(F) + i\beta f R'(F) + O(\frac{F\beta^2 f^2}{|s|^\varepsilon}), \tag{3.24}$$

then from (2.13) and (2.14) and (h1), the multipliers $W_\theta(F) - \tilde{W}_\theta(F)$ and $V_{0,\theta}$ have an analytic extension as bounded operators in $|\beta| \le \beta_0$. Hence we get

Corollary 3.4. *For all $0 < F < F_0$ the family of operators $\{\tilde{H}_{0,\theta}(F); |\theta| \le \beta_0\}$ is a self-adjoint analytic family of operators.*

For some technical points of the Section 6 below, we need to introduce an another family of operators on $L^2(\Omega)$, let $\theta \in \mathbb{R}, |\theta| \le \delta E$ and

$$H_{0,\theta} = U_\theta(T_s + T_u)U_\theta^{-1} = T_{s,\theta} + T_u, \tag{3.25}$$

Note that $H_{0,\theta} \ne H_{0,\theta}(F = 0)$. Indeed the distortion is supported in $\{|s| \ge c'/F\}$ for some $c' > 0$, thus *at least formally* $H_{0,\theta}(F = 0) = H_0$.
Under our assumptions (h1) and (h2). Following arguments of [2, Proposition (3.1)] evoqued above, there exist $F_0 > 0$ and $0 < \beta_0 \le \min\{\delta E, a_0\}$ such that for all $0 < F < F_0$, $\{H_{0,\theta}; |\theta| \le \beta_0\}, D(H_{0,\theta}) = D(H_0)$ is also a self-adjoint analytic family of operators.
In fact the main property of $H_{0,\theta}$ we use in the Section 6 below is

Proposition 3.5. *Suppose (h1-3) hold. Then there exists $0 < \beta_0$ and $0 < F_0$ such that for $0 < F \le F_0$, $\{H_{0,\theta}; \beta = \mathrm{Im}\theta \ge 0, |\theta| \le \beta_0 \le \min\{\delta E, a_0\}\}$ is a family of sectorial operators with a sector contained in*

$$\mathcal{S} = \{z \in \mathbb{C}, -2c\beta \le \arg(z - \lambda_0 + \zeta) \le 0\},$$

for some strictly positive constant c. Here ζ is an error term with $\mathrm{Re}\zeta, \mathrm{Im}\zeta \ge 0$ and $|\zeta| = O(\beta F^2)$.

Proof. We may suppose $\theta = i\beta, 0 \le \beta \le \beta_0$. Note first that for β small enough then

$$\gamma_\theta(s) = \gamma(s + i\beta f(s)) = \gamma(s) + i\beta f\gamma'(s) + O(\frac{\beta^2 f^2}{|s|^\varepsilon}), \tag{3.26}$$

By using (3.22) we have for $\varphi \in D(H_0)$, $\|\varphi\| = 1$,

$$((H_{0,\theta} - \lambda_0)\varphi, \varphi) = (G_\theta \partial_s\varphi, \partial_s\varphi) + (S_\theta\varphi, \varphi) + ((T_u - \lambda_0)\varphi, \varphi) \tag{3.27}$$

where $G_\theta = (1 + \theta f')^{-2} g_\theta$. Consider $q := (G_\theta \partial_s\varphi, \partial_s\varphi) - ((T_u - \lambda_0)\varphi, \varphi)$. By using (h2), we have for F and β small,

$$\begin{aligned} \mathrm{Re}G_\theta^{-1} &= ((1 + u\mathrm{Re}\gamma_\theta)^2 - (u\mathrm{Im}\gamma_\theta)^2)(1 - \beta^2 f'^2) - 4\beta f' u\mathrm{Im}\gamma_\theta(1 + \mathrm{Re}\gamma_\theta) \\ &\ge (1 + u\mathrm{Re}\gamma_\theta)^2(1 - O(\beta^2)) + O(\beta^2 F^{\varepsilon_1}) \ge c_1, \end{aligned} \tag{3.28}$$

for some constant $c_1 > 0$ and then $\mathrm{Re}q \geq c_1 \|\partial_s \varphi\|$. In the other hand,

$$\mathrm{Im}G_\theta^{-1} = 2\big(\beta f'\big((1+u\mathrm{Re}\gamma_\theta)^2 - (u\mathrm{Im}\gamma_\theta)^2\big) + u\mathrm{Im}\gamma_\theta(1+\mathrm{Re}\gamma_\theta)(1-\beta^2 f'^2)\big). \quad (3.29)$$

Since $f' \geq 0$ and by (h3), $\mathrm{Im}\gamma_\theta \geq 0$ then for β and F small $ImG_\theta^{-1} \geq 0$, hence $\mathrm{Im}q \leq 0$. In the other hand in view of (3.26), it is straightforward to check that for F and β small enough there exist $c_2 > 0$ such that $|\mathrm{Im}q| \leq 2\beta c_2 \|\partial_s \varphi\|$. Then by (3.28),

$$q \in \ \{z \in \mathbb{C}, -2c\beta \leq \arg(z - \lambda_0) \leq 0\} \quad (3.30)$$

for some stricly positive constant c. But we know that $(S_\theta \varphi, \varphi) = O(\beta F^2)$, then (3.27) together with (3.30) conclude the proof of the Proposition 3.26. □

Remark 3.6. *The main point in the proof of the Propositions 3.5 and 4.7 below, is the fact that* $\mathrm{Im}\gamma_\theta \geq 0$ *which is insured by (h3). In view of* (3.26), *then a necessary condition to satisfy this condition for* β *small is given by*

$$f(s)\gamma' \geq 0.$$

This means that the curvature has to satisfy $\gamma' \leq 0$ *in a neighbourhood of* $s = -\infty$ *and* $\gamma' \geq 0$ *in a neighbourhood of* $s = \infty$.
Roughly speaking this last inequality is a geometrical non trapping estimate, in the spirit of those given in [3] for the case of electric perturbations.

4 Spectral estimates

The main result in this section is the following. Let $\theta = i\beta$. $f^\sharp = \Phi - 1$ where Φ defined above in the Section 2, clearly $f^\sharp < f'$. Set $\mu_\theta = 1 + \theta f^\sharp$ and

$$\nu_\theta = \left\{ z \in \mathbb{C}, \ \mathrm{Im}\mu_\theta^2(E_- + \lambda_0 - z) < \beta\frac{\delta E}{4} \right\}, \quad (4.31)$$

In the sequel we denote by ${\nu_\theta}^c$ the complement set of ν_θ. Let $\rho(H_\theta(F))$ be the resolvent set of $H_\theta(F)$. Consider first $\theta = i\beta$ then

Proposition 4.7. *Let* $E < 0$ *be an reference energy. Suppose (h1-3). Then there exists* $0 < \beta_0 \leq \min\{\delta E, a_0\}$ *and* $F_0 > 0$ *such that for* $0 < \beta \leq \beta_0$*, and* $0 < F \leq F_0$,

(i) $\nu_\theta \subset \rho(\tilde{H}_{0,\theta}(F))$

(ii) For $z \in \nu_\theta$, $\|(\tilde{H}_{0,\theta}(F) - z)^{-1}\| \leq \mathrm{dist}^{-1}(z, \nu_\theta^c)$.

Proof. Note that in view of (3.21), we have

$$\mu_\theta T_{s,\theta} \mu_\theta = T_1(\theta) + iT_2(\theta) + \mu_\theta \left(T_{s,\theta}\mu_\theta\right),$$

where $T_1(\theta) = -\partial_s \mathrm{Re}\{\mu_\theta^2(1+\theta f')^{-2} g_\theta\}\partial_s$ and $T_2(\theta) = -\partial_s \mathrm{Im}\{\mu_\theta^2(1+\theta f')^{-2} g_\theta\}\partial_s$. $T_1(\theta), T_2(\theta)$ are symmetric operators, let us check that under our assumptions $T_2(\theta)$ is actually negative.
It then sufficient to show that $q' := \mathrm{Im}\mu_\theta^2(1-i\beta f')^2(1+u\bar{\gamma}_\theta)^2 \leq 0$. We have

$$\begin{aligned} q' =& 2\beta\left(f^\sharp - f'\right)\left(1-\beta^2 f' f^\sharp\right)\left((1+u\mathrm{Re}\gamma_\theta)^2 - u^2 \mathrm{Im}{\gamma_\theta}^2)\right) \\ & - 2u\mathrm{Im}\gamma_\theta(1+u\mathrm{Re}\gamma_\theta)\left((1-\beta^2(f^\sharp)^2)(1-\beta^2(f')^2) + 4\beta^2 f^\sharp f'\right) \end{aligned}$$

By using (3.26) together with (h1), for F and β sufficiently small we have

$$\left(1-\beta^2 f' f^\sharp\right)\left((1+u(\mathrm{Re}\gamma_\theta)^2 - u^2\mathrm{Im}{\gamma_\theta}^2)\right) \geq 0$$

and

$$(1-\beta^2(f')^2)(1-\beta^2(f^\sharp)^2) + 4\beta^2 f^\sharp f' \geq 0.$$

We know that $f^\sharp < f'$, therefore, in view of (h3) we get our claim.
In the other hand we have

$$\begin{aligned} \mathrm{Im}\mu_\theta^2\left(\tilde{W}_\theta(F) - E_-\right) &= \left(1-\beta^2 {f^\sharp}^2\right)\mathrm{Im}\tilde{W}_\theta(F) + 2\beta f^\sharp\left(\mathrm{Re}\tilde{W}_\theta(F) - E_-\right) \\ &= -\beta\left((1-\beta^2(f^\sharp)^2)\Phi + 2(1-\Phi)\left(\tilde{W}(F) - E_-\right)\right). \end{aligned}$$

To estimate the r.h.s. of this expression, we note that from the definition of $\tilde{W}(F)$, if $s \in \mathrm{supp}(1-\Phi)$, $\tilde{W}(F) - E_- > \delta E + O(F)$. Accordingly for F and β small,

$$\mathrm{Im}\mu_\theta^2\left(\tilde{W}_\theta(F) - E_-\right) \leq -\beta\frac{\delta E}{2} \tag{4.32}$$

Thus we get

$$\begin{aligned} \mathrm{Im}\mu_\theta\left(\tilde{H}_{0,\theta}(F) - z\right)\mu_\theta = \mathrm{Im}\mu_\theta(T_{s,\theta}\mu_\theta) + T_2(\theta) + \mathrm{Im}\mu_\theta^2 T_u + \mathrm{Im}\mu_\theta^2\left(\tilde{W}_\theta(F) - E_-\right) + \\ \mathrm{Im}\mu_\theta^2(E_- - z) \leq \mathrm{Im}\mu_\theta(T_{s,\theta}\mu_\theta) - \beta\frac{\delta E}{2} + \mathrm{Im}\mu_\theta^2(E_- + \lambda_0 - z) \end{aligned}$$

and since

$$\mathrm{Im}\mu_\theta(T_{s,\theta}\mu_\theta) = O(\beta^2 F), \tag{4.33}$$

then for $z \in \nu_\theta$, for F and β small enough

$$\mathrm{Im}\mu_\theta\left(\tilde{H}_{0,\theta}(F) - z\right)\mu_\theta \leq -\beta\frac{\delta E}{4} + \mathrm{Im}\mu_\theta^2(E_- + \lambda_0 - z) < 0.$$

Thus the proof of the proposition follows (see [3] or [7] for more details). $\square$

By standard arguments the Proposition 4.7 holds for $0 < |\theta| \leq \beta_0, 0 < \mathrm{Im}\theta$ and $0 < F < F_0$.

5 Meromorphic continuation of the resolvent

Under conditions of the Proposition 4.7. Set $V_\theta = V_{0,\theta} + W_\theta(F) - \tilde{W}_\theta(F)$. Introduce the following operator, let $\theta \in \mathbb{C}$, $|\theta| < \beta_0$, $0 < F \leq F_0$, $z \in \nu_\theta$ and

$$K_\theta(F, z) = V_\theta(\tilde{H}_{0,\theta}(F) - z)^{-1}. \tag{5.34}$$

Then

Proposition 5.8. *Suppose (h1-3). Then there exists* $0 < \beta_0 \leq \min\{\delta E, a_0\}$ *and* $F_0 > 0$ *such that for* $0 < |\theta| \leq \beta_0, \mathrm{Im}\theta = \beta > 0$, *and* $0 < F \leq F_0$,

(i) $z \in \nu_\theta \to K_\theta(F, z)$ *is an analytic compact operator valued function.*

(ii) *For* $z \in \nu_\theta$, $Imz > 0$ *large enough,* $\|K_\theta(F, z)\| < 1$.

Proof. Let us first show that under these conditions, $K_\theta(F, z), \theta = i\beta, \mathrm{Im}z > 0$ are compact operators, this allows to prove the Proposition 5.8 i). We know from (3.24) and (h1-2) that $V_\theta = O(\frac{F}{|s|^{\epsilon-1}})$ as $s \to \pm\infty$. Denote by $\mathbb{I}_{\mathcal{H}}$ the identity operator on the space $\mathcal{H}$. Let $h_0 = -\partial_s^2 \otimes \mathbb{I}_{L^2(0,d)} + \mathbb{I}_{L^2(\mathbb{R})} \otimes T_u$. Then the decay property of V_θ imply that the operator $V_\theta(h_0 - z)^{-1}$ is compact for $\mathrm{Re}z < \lambda_0$ (see e.g. [2] or [8])

We have

$$\tilde{H}_{0,\theta}(F) - h_0 = \partial_s(1 - G_\theta)\partial_s + S_\theta + \tilde{W}_\theta(F) \tag{5.35}$$

where G_θ, S_θ are given respectively by (3.27), (3.23) and $\tilde{W}_\theta(F)$ is defined in the Section 3. Hence,

$$K_\theta(F, z) = V_\theta(h_0 - z)^{-1} + V_\theta(h_0 - z)^{-1}\big(\tilde{W}_\theta(F) + \partial_s(1 - G_\theta)\partial_s + S_\theta\big)(\tilde{H}_{0,\theta}(F) - z)^{-1}. \tag{5.36}$$

Let first show that the operator $I_1 := V_\theta(h_0 - z)^{-1}\tilde{W}_\theta(F)(\tilde{H}_{0,\theta}(F) - z)^{-1}$ is compact. This follows from the Herbst's argument [8]. Indeed let $l(s) := (1 + |s|^2)^{1/2}$, then

$$I_1 = V_\theta l(h_0 - z)^{-1}\frac{\tilde{W}_\theta(F)}{l}(\tilde{H}_{0,\theta}(F) - z)^{-1} + V_\theta(h_0 - z)^{-1}[l, h_0](h_0 - z)^{-1}\frac{\tilde{W}_\theta(F)}{l}(\tilde{H}_{0,\theta}(F) - z)^{-1}. \tag{5.37}$$

Where $[A, B]$ denotes the commutator of the operators A and B. The operator $V_\theta l(h_0 - z)^{-1}$ is compact since $V_\theta(s, u)l(s) = O(\frac{F}{|s|^{\varepsilon-3}})$ as $s \to \pm\infty$ and $\varepsilon > 3$. This holds true for the first operator of the r.h.s. of (5.37) since $\frac{\tilde{W}_\theta(F)}{l}(\tilde{H}_{0,\theta}(F) - z)^{-1}$ is a bounded operator. In the other hand by the closed graph theorem, $[l, h_0](h_0 - z)^{-1}$ is a bounded operator. So it follows that the second operator of of the r.h.s. of (5.37) and then I_1 is also compact.

Set $I_2 = V_\theta(h_0 - z)^{-1}\partial_s(1-G_\theta)\partial_s(\tilde{H}_{0,\theta} - z)^{-1}$. Let us show $\partial_s(1-G_\theta)\partial_s(\tilde{H}_{0,\theta}(F) - z)^{-1}$, $\mathrm{Im}z > 0$ is a bounded operators. In view of the Corollary 3.4, we are left to show that $\partial_s(1 - G_\theta)\partial_s(\tilde{H}_0(F) - z)^{-1}$ is bounded. We have

$$\partial_s(1-G_\theta)\partial_s(\tilde{H}_0(F)-z)^{-1} = \partial_s(1-G_\theta)\partial_s(H_0-z)^{-1} + \partial_s(1-G_\theta)\partial_s(H_0-z)^{-1}\tilde{W}(F)(\tilde{H}_0(F)-z)^{-1}.$$

By the closed graph theorem $\partial_s(1 - G_\theta)\partial_s(H_0 - z)^{-1}$ is bounded. Now the second term of the r.h.s. of this equality can be written as

$$\partial_s(1 - G_\theta)\partial_s l(H_0 - z)^{-1}\frac{\tilde{W}(F)}{l}(\tilde{H}_0(F) - z)^{-1} + \\ \partial_s(1 - G_\theta)\partial_s(H_0 - z)^{-1}[H_0, l](H_0 - z)^{-1}\frac{\tilde{W}(F)}{l}(\tilde{H}_0(F) - z)^{-1}. \quad (5.38)$$

Notice that under (h2) then $(1 - G_\theta)l$ as well as $(1 - G_\theta)l'$ are bounded functions. Hence following the same arguments as above we are done. Evidently $I_3 = V_\theta(h_0 - z)^{-1}S_\theta(\tilde{H}_{0,\theta} - z)^{-1}$ is also a compact operator. This proves our claim.

In the other hand the Proposition 4.7 implies ii). $\square$

We now prove the first part of the Theorem 1.2.

Let $0 < F \leq F_0$ and $E < 0$. For $0 < |\theta| < \beta_0, \mathrm{Im}\theta > 0$ by the Proposition 4.7, the Lemma 5.8 and the Fredholm alternative theorem, the operator $\mathbb{I}_{L^2(\Omega)} + K_\theta(F, z)$ is invertible for all $z \in \nu_\theta \setminus \mathcal{R}$ where $\mathcal{R}$ is a discrete set. In the bounded operator sense, we have

$$(H_\theta(F) - z)^{-1} = (\tilde{H}_{0,\theta}(F) - z)^{-1}\big(\mathbb{I}_{L^2(\Omega)} + K_\theta(F, z)\big)^{-1}. \quad (5.39)$$

This implies that $\nu_\theta \setminus \mathcal{R} \subset \rho(H_\theta(F))$.

Choose β_0 so small that there exists a dense subset of analytic vectors associated to the transformation U_θ in $|\theta| < \beta_0$ (see [2, Remark (3.3)]). We denote this set by $\mathcal{A}$. Then standards arguments of the distortion theory, and (5.39) imply that for all $\varphi \in \mathcal{A}$

$$\mathcal{R}_\varphi(z) = \big((H(F) - z)^{-1}\varphi, \varphi\big), \mathrm{Im}z > 0 \quad (5.40)$$

has an meromophic extension in ν_θ given by

$$\mathcal{R}_\varphi(z) = \big((\tilde{H}_{0,\theta}(F) - z)^{-1}\big(\mathbb{I}_{L^2(\Omega)} + K_\theta(F, z)\big)^{-1}\varphi_\theta, \varphi_{\bar{\theta}}\big)$$

We define the resonances of the operator $H_\theta(F)$ as the poles of $\mathcal{R}_\varphi$. They are locally θ-independent and in view of (5.39) they are the discrete eigenvalues of the operator $H_\theta(F)$. $\square$

6 Resonances

We want to prove the following result, let $\theta = i\beta$, $0 < \beta \leq \beta_0$, β_0 as in the Proposition 5.8 and the proof of the Theorem 1.2 i). Then

Proposition 6.9. *Let E_0 is a negative eigenvalue of H with multiplicity n. There exists $F_0 > 0$ such that for $0 < F \leq F_0$,$H_\theta(F)$ has exactly n eigenvalues denoted by $Z_0, ..., Z_{n-1}$, satisfying $\lim_{F\to 0} |Z_j - E_0| = 0$.*

The Proposition 6.9 implying the Theorem 1.2 ii).

Proof. Following [2, Section 5], to prove the Proposition 6.9 we only have to prove that under (h1-3), for $E < 0$, $\theta = i\beta, 0 < \beta \leq \beta_0$. Then

$$\|K_\theta(F, z) - K(z)\| \to 0 \text{ as } F \to 0 \tag{6.41}$$

uniformly in $z \in \mathcal{K}$, where $\mathcal{K}$ is a compact subset of $\nu_\theta \bigcap \rho(H_0)$. By a continuity argument it is sufficient to prove that for some $z_0 \in \mathcal{K}$. We have

$$\begin{aligned} K_\theta(F, z_0) - K(z_0) =& (V_{0,\theta} - V_0)(H_0 - z_0)^{-1} + (W_\theta(F) - \tilde{W}_\theta(F))(\tilde{H}_{0,\theta}(F) - z_0)^{-1} + \\ & V_{0,\theta}((\tilde{H}_{0,\theta}(F) - z_0)^{-1} - (H_0 - z_0)^{-1}). \end{aligned} \tag{6.42}$$

From (6.42), the proof needs several steps. First we consider the two first term of the r.h.s. of (6.42). In view of (6.42) and (3.19), we can see that

$$\|V_{0,\theta} - V_0\|_\infty \leq C\big(\|\gamma_\theta - \gamma\|_\infty + \|\gamma_\theta' - \gamma'\|_\infty + \|\gamma_\theta^{''} - \gamma^{''}\|_\infty\big)$$

for some constant $C > 0$. Hence we can estimate each term of the r.h.s. of this last inequality by using Taylor expansions w.r.t. θ (see e.g. (3.26). Then we find

$$\|\gamma_\theta - \gamma\|_\infty, \|\gamma_\theta' - \gamma'\|_\infty, \|\gamma_\theta^{''} - \gamma^{''}\|_\infty = O(\beta F^{\varepsilon - 1})$$

and then $\|(V_{0,\theta} - V_0)(H_0 - z_0)^{-1}\| \to 0$ as $F \to 0$. The Proposition 4.7 and (3.24) imply that $\|(W_\theta(F) - \tilde{W}_\theta(F))(\tilde{H}_{0,\theta}(F) - z_0)^{-1}\| \to 0$ as $F \to 0$. Hence we are left to show that this is true for the third term of the r.h.s of (6.42) i.e.

$$\delta H(z_0) := V_{0,\theta}((\tilde{H}_{0,\theta}(F) - z_0)^{-1} - (H_0 - z_0)^{-1}).$$

Let $H_{0,\theta}$ be the operator introduced in the Section 3. We use the estimate

$$\begin{aligned} \|\delta H(z_0)\| \leq & \|V_{0,\theta}((H_{0,\theta} - z_0)^{-1} - (\tilde{H}_{0,\theta}(F) - z_0)^{-1})\| + \\ & \|V_{0,\theta}((H_{0,\theta} - z_0)^{-1} - H_0 - z_0)^{-1})\|. \end{aligned} \tag{6.43}$$

First we consider

$$\delta_1 H(z_0) = V_{0,\theta}(H_{0,\theta} - z_0)^{-1}\tilde{W}_\theta(F)(\tilde{H}_{0,\theta}(F) - z_0)^{-1}.$$

Following arguments of the proof of the Proposition (5.8) we have

$$\begin{aligned}\delta_1 H(z_0) =& V_{0,\theta} l (H_{0,\theta}-z_0)^{-1}\frac{\tilde{W}_\theta(F)}{l}(\tilde{H}_{0,\theta}(F)-z_0)^{-1}+\\ & V_{0,\theta}[(H_{0,\theta}-z_0)^{-1},l]\frac{\tilde{W}_\theta(F)}{l}(\tilde{H}_{0,\theta}(F)-z_0)^{-1}.\end{aligned} \tag{6.44}$$

Clearly $\|\frac{\tilde{W}_\theta(F)}{l}\| \to 0$ as $F \to 0$. In the other hand we know that $V_{0,\theta}, V_{0,\theta}l$ are bounded uniformly w.r.t. F. In view of the Propositions 3.5 and 4.7, this also holds for the resolvents $(H_{0,\theta}-z_0)^{-1}$ and $(\tilde{H}_{0,\theta}(F)-z_0)^{-1}$. Let us show that this is again true for $[(H_{0,\theta}-z_0)^{-1},l]$ then this will imply that $\|\delta_1 H(z_0)\| \to 0$ as $F \to 0$.
We have,

$$[(H_{0,\theta}-z_0)^{-1},l] = -(H_{0,\theta}-z_0)^{-1}((G_\theta l')'+2l'G_\theta\partial_s)(H_{0,\theta}-z_0)^{-1}.$$

and $\mathrm{Re}G_\theta \geq 0, \mathrm{Im}G_\theta \leq 0$. It is then sufficient to show that $\|(\mathrm{Re}G_\theta)^{1/2}\partial_s(H_{0,\theta}-z_0)^{-1}\|$ and $\|(-\mathrm{Im}G_\theta)^{1/2}\partial_s(H_{0,\theta}-z_0)^{-1}\|$ are uniformly bounded w.r.t. F for F small. Recall that $H_{0,\theta} = -\partial_s G_\theta \partial_s + T_u + S_\theta$. Then our last claim follows from the arguments evoked above and that for $\varphi \in L^2(\Omega)$, $\|\varphi\| = 1$,

$$\begin{aligned}\|(\mathrm{Re}G_\theta)^{1/2}\partial_s(H_{0,\theta}-z_0)^{-1}\varphi\| \leq & \mathrm{Re}\big((H_{0,\theta}-z_0)^{-1}\varphi,\varphi)\big)\\ & -\mathrm{Re}\big((H_{0,\theta}-z_0)^{-1}\varphi,(S_\theta+z_0)(H_{0,\theta}-z_0)^{-1}\varphi\big)\end{aligned}$$

and

$$\begin{aligned}\|(-\mathrm{Im}G_\theta)^{1/2}\partial_s(H_{0,\theta}-z_0)^{-1}\varphi\| = & -\mathrm{Im}\big((H_{0,\theta}-z_0)^{-1}\varphi,\varphi)\big)\\ & +\mathrm{Im}\big((H_{0,\theta}-z_0)^{-1}\varphi,(S_\theta+z_0)(H_{0,\theta}-z_0)^{-1}\varphi\big).\end{aligned}$$

Evidently this imply that there exists a constant $c' > 0$ s.t.

$$\|(\mathrm{Re}G_\theta)^{1/2}\partial_s(H_{0,\theta}-z_0)^{-1}\|, \|(-\mathrm{Im}G_\theta)^{1/2}\partial_s(H_{0,\theta}-z_0)^{-1}\| \leq \|(H_{0,\theta}-z_0)^{-1}\|+c'\|(H_{0,\theta}-z_0)^{-1}\|^2.$$

Now consider

$$\delta_2 H(z_0) = V_{0,\theta}(H_{0,\theta}-z_0)^{-1}(T_s-T_{s,\theta})(H_0-z_0)^{-1}.$$

For F and β small, $T_s - T_{s,\theta} = \partial_s G \partial_s + S_\theta$ with $G = (\gamma_\theta-\gamma)G_1 + i\beta f' G_2$ where G_1, G_2 are uniformly bounded functions w.r.t. F (see e.g. (1.4) and (3.22).
Moreover we know from (3.26) that $\gamma_\theta - \gamma = O(\beta F^{\varepsilon-1})$, and $S_\theta = O(\beta F^2)$ then

$$\begin{aligned}&\|V_{0,\theta}(H_{0,\theta}-z_0)^{-1}(\partial_s(\gamma_\theta-\gamma)G_1\partial_s+S_\theta)(H_0-z_0)^{-1}\| \leq \\ & C\beta\big(F^\varepsilon\|(H_{0,\theta}-z_0)^{-1}\|\|\partial_s G_3\partial_s(H_0-z_0)^{-1}\|+\\ & \qquad F^2\|(H_{0,\theta}-z_0)^{-1}\|\|(H_0-z_0)^{-1}\|\big)\end{aligned} \tag{6.45}$$

for some constant $C > 0$. Where G_3 is uniformly bounded w.r.t. F. So this term vanishes as $F \to 0$.

To study the second term, we use a different strategy. We have

$$V_{0,\theta}(H_{0,\theta} - z_0)^{-1}\partial_s f' G_2 \partial_s (H_0 - z_0)^{-1} = \\ V_{0,\theta}(H_0 - z_0)^{-1}(H_0 - z_0)(H_{0,\theta} - z_0)^{-1}\partial_s f' G_2 \partial_s (H_0 - z_0)^{-1} \tag{6.46}$$

We know that the operator $V_{0,\theta}(H_0 - z)^{-1}$ is a compact operator (see e.g. the proof of the Lemma 5.8) then to prove that the operator in the l.h.s. of (6.46) converges in the norm sense to $0_{\mathcal{B}(L^2(\Omega))}$ as $F \to 0$, it is sufficient to show that $(H_0 - z_0)(H_{0,\theta} - z_0)^{-1}\partial_s f' G_2 \partial_s (H_0 - z_0)^{-1}$ converges strongly to $0_{\mathcal{B}(L^2(\Omega))}$ as $F \to 0$.

Recall that $\mathcal{C} = \{\varphi = \tilde{\varphi}\lfloor_\Omega, \tilde{\varphi} \in C_0^\infty(\mathbb{R}^2); \tilde{\varphi}\lfloor_{\partial\Omega} = 0\}$ is a core of H_0, thus for $z \in \rho(H_0)$, $\mathcal{C}' = (H_0 - z_0)\mathcal{C}$ is dense in $L^2(\Omega)$. Set $\psi = (H_0 - z_0)\varphi$, $\varphi \in \mathcal{C}$.

Since the field f is choosed s.t. f' has support contained in $|s| > c'/F$ for some $c' > 0$ then $\lim_{F\to 0} \|\partial_s f' G_2 \partial_s \varphi\| = 0$. By using standard arguments of the perturbation theory and the proposition 3.5, for F small, the operator $(H_0 - z_0)(H_{0,\theta} - z_0)^{-1}$ has a norm which is uniformly bounded w.r.t F. This proves our claim on $\mathcal{C}'$.

In the other hand $\partial_s f' G_2 \partial_s (H_0 - z_0)^{-1}$ is bounded operator with a norm uniformly bounded w.r.t. F, for F small. Then the strong convergence follows. □

7 Concluding remarks

In this last section we would like to give some remarks about the field regime related to this problem. Let us mention that the first result was given by P. Exner in [5], for $\eta = \frac{\pi}{2}$ and $\alpha_0 = 0$. But P. Exner did not consider the question of existence of resonances.

This issue was addressed by us in [2]. In this paper we have considered the conditions $|\eta| < \frac{\pi}{2}$ and $|\eta - \alpha_0| < \frac{\pi}{2}$. Roughly speaking this corresponds to the classical picture of the Stark effect for one dimensional Schrödinger operators with local potential i.e. the field interaction $W(F) \to -\infty$ as $s \to -\infty$ and $W(F) \to \infty$ as $s \to \infty$. For any negative reference energy the non trapping region coinciding with a neighbourhood of $s = -\infty$. In this case we prove an analog of the Theorem 1.2.

Evidently the regime $|\eta| > \frac{\pi}{2}$ and $|\eta - \alpha_0| > \frac{\pi}{2}$ is a symmetric to the above mentioned case.

Suppose now that $|\eta| > \frac{\pi}{2}$ and $|\eta - \alpha_0| < \frac{\pi}{2}$. Then from (2.9) and (2.10), clearly $W(F) \to \infty$ as $s \to \pm\infty$ i.e. it is a confining potential. By using standard arguments (see e.g. [13]) it easy to see that $H(F)$ has a compact resolvent and then only discrete spectrum.

Indeed consider the following operator in $L^2(\Omega)$. Let $F > 0$,

$$h(F) = T_s + T_u + w(F).$$

where $w(F) = F\cos(\eta)s$ if $s \le 0$ and $w(F) = F\cos(\eta - \alpha_0)s$ if $s > 0$. Then by (h1) and (h2), there exists a strictly positive constant c such that in the form sense we have

$$h(F) \ge h_1(F) := (-c\partial_s^2 + w(F)) \otimes \mathbb{I}_u + \mathbb{I}_s \otimes -\partial_u^2. \tag{7.47}$$

But the operator $-\partial_s^2 + w(F)$ has a compact resolvent [13]. Let $\{p_n, n \ge 0\}$ the eigen-projectors corresponding to the transverse modes $\{\lambda_n, n \ge 0\}$. Since

$$(h_1(F) + 1)^{-1} = \oplus_{n\ge 1}(-c\partial_s^2 + w(F) + \lambda_n + 1)^{-1}) \otimes p_n$$

Then $(h_1(F) + 1)^{-1}$ is a compact operator as a norm limit of compact operators. Hence $h_1(F)$ satisfies the Rellich criterion [13] and in view of (7.47) it is also true for $h(F)$ so $(h(F)+1)^{-1}$ is compact. Since $H(F) - h(F)$ is bounded then $(H(F)+1)^{-1}$ is also a compact operator.

References

[1] J. Aguilar, J.M. Combes, A class of analytic perturbations for monebody Schrödinger Hamiltonians, *Commun. Math. Phys.*, 22 (1971).

[2] P. Briet, M. Gharsalli, Stark resonances in 2-dimensional curved quantum waveguides, *Report, Math. Phys.* 9 (2015).

[3] P. Briet: General estimates on distorted resolvents and application to Stark hamiltonians, *Rev. Math. Phys.* 8, no. 5 (1996).

[4] P. Duclos and P. Exner: curvature induced bound states in quantum waveguides in two and three dimensions, *Rev. Math. Phys.*, **7**, 1, (1995).

[5] P. Exner, Quantum pipette, *J. Phys. A Math. Gen.*, **28**, (1995).

[6] P. Exner and P. Seba, Bound states in curved quantum waveguides, *J. Math. Phys.*, Vol, 30, (1989).

[7] M. Gharsalli, *Analyse spectrale et résonances*, P.H.D. Thesis, Univ. El Manar, Tunis, 2016.

[8] I. Herbst, Dilation analyticity in constant electric field, *Comm. Math. Phys.*, 64 (1979).

[9] P. Hislop, I.M. Sigal *Introduction to spectral theory with applications to Schrödinger operators* New York : Springer Verlag, 1996.

[10] W. Hunziker, Distorsion analyticity and molecular resonance curves, *Ann. Inst. H. Poincaré: Phys.Théor.* 45 (1986).

[11] T. Kato: *Perturbation theory for Linear Operators*, 2nd Edition, Springer Verlag, Berlin, Heilderberg, 1995.

[12] J. Kriz: *Spectral properties of planar quantum waveguides with combined boundary conditions*, P.H.D. Thesis, Charles University Prague, 2003.

[13] M. Reed; B. Simon, *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York,1978.